An optimal policy for partially observable Markov decision processes with non-independent monitors

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Abstract

Purpose – This research investigated the optimal structure of a discrete-time Markov deterioration system monitored by multiple non-independent monitors. The purpose is to obtain a sufficient condition with which the optimal policy is given by a control limit policy.

Design/methodology/approach – The model of this research is formulated as a partially observable Markov decision process. The problem is to obtain an optimal policy which can minimize the expected total discounted cost over an infinite horizon.

Findings – The research found that the expected optimal cost function over an infinite horizon has a property of control limit policy given the conditions that a transition probability having a property of totally positive of order 2 and a conditional probability of the monitors having a property of weak multivariate monotone likelihood ratio. Furthermore, we showed that the optimal policy has only four action regions at most.

Practical implications – If the optimum policy can be limited to a control limit policy, the tremendous amount of calculation time required to find the optimum procedure can be reduced. This enables the best decision to be identified in a much shorter period of time.

Originality/value – A deterioration system monitored incompletely by one monitor has been studied in the previous research. This research considered the case of a multiple number monitors whose observations were not independent.

Keywords Condition monitoring, Optimization techniques, Failure (mechanical), Reliability management

Paper type Research paper

1. Introduction

Breakdowns in large, complex systems can seriously affect society. Preventive maintenance plays an important role in avoiding such breakdowns. Many kinds of maintenance problems of various systems have been studied using theories of reliability and maintainability. Vaidyanathan *et al.* (2002), for example, investigated a continuous time queuing model for an inspection-based preventive maintenance problem. Derman (1963) studied an optimal replacement problem for discrete-time Markov deterioration systems in which the state of the system is completely identified at any given time. Rosenfield (1976) and White (1978) discussed optimal inspection and replacement problems under the assumption that the system's state can be observed only through costly inspection. Ohnishi *et al.* (1986) explored a system monitored incompletely by one monitoring mechanism. However, the reliability of a system using only one monitor is degraded by two types of contradictory failures, "false alarms" and "failure to alarm". Taking this into account, we investigated a system monitored by

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several monitors whose observations were not independent and determined under what conditions there exists an optimal control limit policy.

2. Model

2.1 Description

In our system, the internal true state cannot be observed directly. Let X denote the true state, and its value comes from a finite set $\{1, 2, \ldots, n\}$ in which the numbers are ordered to reflect the degree of system deterioration. That is, state 1 denotes the best state, i.e. the system is like new, and state n denotes the most deteriorated state. The state deteriorates based on a stationary discrete-time Markov chain having a known transition law. Let $\boldsymbol{P}$ be the transition probability matrix, in which element p_{ij} denotes the one-step transition probability from state i to state j. At each time period, the state is monitored incompletely by monitors that give information related to the true state of the system. We assume there are $L(\geq 1)$ monitors. The outcomes of the L monitors are given as $\boldsymbol{M} = (M^{(1)}, \ldots, M^{(k)}, \ldots, M^{(L)})$, where $M^{(k)}$ denotes the outcome of the kth monitor and comes from a finite set $\{1, 2, \ldots, m_k\}$. Let

$$\Gamma = \begin{pmatrix} \gamma_{1(1,\ldots,1)} & \cdots & \gamma_{1(\theta_1,\ldots,\theta_L)} & \cdots & \gamma_{1(m_1,\ldots,m_L)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \gamma_{j(1,\ldots,1)} & \cdots & \gamma_{j(\theta_1,\ldots,\theta_L)} & \cdots & \gamma_{j(m_1,\ldots,m_L)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \gamma_{n(1,\ldots,1)} & \cdots & \gamma_{n(\theta_1,\ldots,\theta_L)} & \cdots & \gamma_{n(m_1,\ldots,m_L)} \end{pmatrix}$$

be the conditional probability matrix that describes the relationships between the system's true states and the monitors' outcomes.

$$\gamma_{j(\theta_1,\ldots,\theta_L)} = \Pr(M^{(1)} = \theta_1, \ldots, M^{(k)} = \theta_k, \ldots, M^{(L)} = \theta_L | X = j)$$

Three actions, "keep", "inspect", and "replace", are considered. "Keep" means an action that continues system operation with incomplete monitoring, and the operating cost per period in state i is given by C_i. "Inspect" means an action to operate the system with inspection, and this action reveals the exact state of the system with certainty. Inspection cost $I(> 0)$ is assumed to be constant. Similarly, "replace" means an action to replace the system with a new one, and replacement cost $R(> 0)$ is assumed to be constant. At any given time period, only one of these three actions can be selected as the optimal one.

2.2 Assumptions

A1. Transition probability matrix $\boldsymbol{P}$ has a property of totally positive of order 2 (TP_2, Karlin, 1968), denoted by $\boldsymbol{P} \in TP_2$:

$$\begin{vmatrix} p_{ij} & p_{ij'} \\ p_{i'j} & p_{i'j'} \end{vmatrix} \geq 0 \quad \text{for } i < i', \quad j < j'. \tag{1}$$

A2. Observational probability matrix Γ has a property of weak multivariate monotone likelihood ratio (weak MLR) (Whitt, 1982), denoted by $\Gamma \in$ weak MLR:

$$\begin{vmatrix} \gamma_{i\boldsymbol{\theta}} & \gamma_{i\boldsymbol{\theta}'} \\ \gamma_{j\boldsymbol{\theta}} & \gamma_{j\boldsymbol{\theta}'} \end{vmatrix} \geq 0 \quad \text{for } i < j, \quad \boldsymbol{\theta} \leq \boldsymbol{\theta}', \tag{2}$$

where the vector of monitor information has a partial order: $\boldsymbol{\theta} \leq \boldsymbol{\theta}'$, for $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_L)$ and $\boldsymbol{\theta}' = (\theta'_1, \ldots, \theta'_L)$ if $\theta_k \leq \theta'_k$ for each $k \in \{1, \ldots, L\}$.

A3. C_i is a nondecreasing function of state i.

A4. $I(> 0)$ is constant.

A5. $R(> 0)$ is constant.

A1. means that, as the system deteriorates, it is more likely to make a transition to a higher state. *A2.* implies that higher states of the system give rise to higher outcome levels of the monitoring probabilistically. *A3.* means that, as the system deteriorates, it becomes more costly to operate. *A4.* and *A5.* mean that the inspection and replacement costs of the system are constant.

3. Optimal keep, inspect, and replace problem

At the beginning of any time period, "keep", "inspect", or "replace", is selected as an action. An optimal policy is the sequence of actions that minimizes the total cost incurred in both current and future time periods. Since the state information obtained from monitors is incomplete, the decision maker needs to determine the most suitable action by inferring the system's exact state from the current monitoring information and the system's history. This problem is formulated as partially observable Markov decision processes.

The problem is how to minimize the expected total discounted cost over an infinite horizon. Let $\boldsymbol{\Pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ be the prior state probability vector of X, where

$$\pi_i = \Pr(X = i), \quad \sum_{i=1}^{n} \pi_i = 1, \text{ and } 0 \leq \pi_i \leq 1 \quad \text{for any } i$$

The transition between the states is specified below.

- When "keep" is selected, $\boldsymbol{\Pi} \Rightarrow T(\boldsymbol{\Pi}, \boldsymbol{\theta})$ with probability $P(\boldsymbol{\theta}|\boldsymbol{\Pi})$, where

$$P(\boldsymbol{\theta}|\boldsymbol{\Pi}) = \Pr(M = \boldsymbol{\theta}|\boldsymbol{\Pi}) = \sum_{j=1}^{n}\sum_{i=1}^{n} \pi_i p_{ij} \gamma_{j\boldsymbol{\theta}},$$

$$T_j(\boldsymbol{\Pi}, \boldsymbol{\theta}) = \Pr(X = j|M = \boldsymbol{\theta}, \boldsymbol{\Pi}), \quad = \frac{\sum_{i=1}^{n} \pi_i p_{ij} \gamma_{j\boldsymbol{\theta}}}{\sum_{j=1}^{n}\sum_{i=1}^{n} \pi_i p_{ij} \gamma_{j\boldsymbol{\theta}}},$$

$$T(\boldsymbol{\Pi}, \boldsymbol{\theta}) = (T_1(\boldsymbol{\Pi}, \boldsymbol{\theta}), T_2(\boldsymbol{\Pi}, \boldsymbol{\theta}), \ldots, T_n(\boldsymbol{\Pi}, \boldsymbol{\theta})).$$

Note that the updated probability vector, $\boldsymbol{T}(\boldsymbol{\Pi}, \boldsymbol{\theta})$, is calculated using Bayes' formula.

- When "inspect" is selected, $\boldsymbol{\Pi} \Rightarrow e^j$ with probability

$$\sum_{i=1}^{n} \pi_i p_{ij}$$

since the true state is known, and state probability e^j is given as

$$e^j = (0, \ldots, 0, 1, 0, \ldots, 0),$$

where the jth element is 1.

- When "replace" is selected, $\boldsymbol{\Pi} \Rightarrow e^1$ with probability 1, where

$$e^1 = (1, 0, \ldots, 0, \ldots, 0).$$

4. Properties of optimal total cost function

4.1 Optimal expected total cost function

Let $V(\boldsymbol{\Pi})$ denote the optimal expected total cost function over an infinite horizon with initial state $\boldsymbol{\Pi}$.

$$V(\boldsymbol{\Pi}) = \min \begin{cases} \sum_{i=1}^{n} \pi_i C_i + \beta \sum_{\theta_1=1}^{m_1} \cdots \sum_{\theta_L=1}^{m_L} P(\boldsymbol{\theta}|\boldsymbol{\Pi}) V(\boldsymbol{T}(\boldsymbol{\Pi}, \boldsymbol{\theta})) & \text{(keep)} \\ \sum_{i=1}^{n} \pi_i C_i + I + \beta \sum_{j=1}^{n} \sum_{i=1}^{n} \pi_i p_{ij} V(e^j) & \text{(inspect)} \\ R + \beta V(e^1) & \text{(replace)} \end{cases} \quad (3)$$

This is a recursive function that can be calculated based on initial state $\boldsymbol{\Pi}$, and $\beta (0 < \beta < 1)$ is the discount factor. The three terms on the right correspond to the total expected costs for the current and future periods when the actions "keep", "inspect", and "replace", are selected at the beginning.

Furthermore, the action that minimizes the right side of equation (3) is the optimal action, and it must be selected for state $\boldsymbol{\Pi}$. Hence, an optimal policy is obtained by selecting the action that minimizes the optimal expected total cost for each $\boldsymbol{\Pi}$.

For notational convenience, we define three transformations.

$$KV(\boldsymbol{\Pi}) \triangleq \sum_{i=1}^{n} \pi_i C_i + \beta \sum_{\theta_1=1}^{m_1} \cdots \sum_{\theta_L=1}^{m_L} P(\boldsymbol{\theta}|\boldsymbol{\Pi} V(T(\boldsymbol{\Pi}, \boldsymbol{\theta})) \quad (4)$$

$$IV(\boldsymbol{\Pi}) \triangleq \sum_{i=1}^{n} \pi_i C_i + I + \beta \sum_{j=1}^{n} \sum_{i=1}^{n} \pi_i p_{ij} V(e^j) \quad (5)$$

$$RV(\boldsymbol{\Pi}) \triangleq R + \beta V(e^1), \quad (6)$$

where $KV(\boldsymbol{\Pi})$, $IV(\boldsymbol{\Pi})$, and $RV(\boldsymbol{\Pi})$ correspond to "keep", "inspect", and "replace". We can now write equation (3) as

$$V(\boldsymbol{\Pi}) = \min\{KV(\boldsymbol{\Pi}), IV(\boldsymbol{\Pi}), RV(\boldsymbol{\Pi})\}. \quad (7)$$

4.2 Control limit policy

For prior state probability vectors $\mathbf{\Pi}^1 = (\pi_1^1, \pi_2^1, \ldots, \pi_n^1)$ and $\mathbf{\Pi}^2 = (\pi_1^2, \pi_2^2, \ldots, \pi_n^2)$, if $V(\mathbf{\Pi}^1) \leq V(\mathbf{\Pi}^2)$ holds for $\mathbf{\Pi}^1 <_T \mathbf{\Pi}^2$, we say optimal cost function $V(\mathbf{\Pi})$ exhibits a control limit policy (Figure 1). Here, partial order $<_T$ for the two vectors is defined as: $\mathbf{\Pi}^1 <_T \mathbf{\Pi}^2$ if

$$\frac{\pi_i^1}{\pi_i^2} \geq \frac{\pi_j^1}{\pi_j^2}, \quad \text{for all } i,j\,(1 \leq i < j \leq n), \tag{8}$$

and we say that vector $\mathbf{\Pi}$ has a property of TP_2.

An example of a non-control limit policy is shown in Figure 2. This policy implies that the optimal expected total cost over an infinite horizon with initial state $\mathbf{\Pi}$ is a monotonically nondecreasing function of $\mathbf{\Pi}$ with respect to the partial order "TP_2". That is, the more deteriorated the initial state of the system in a probabilistic sense of TP_2 order, the larger the cost incurred in the future.

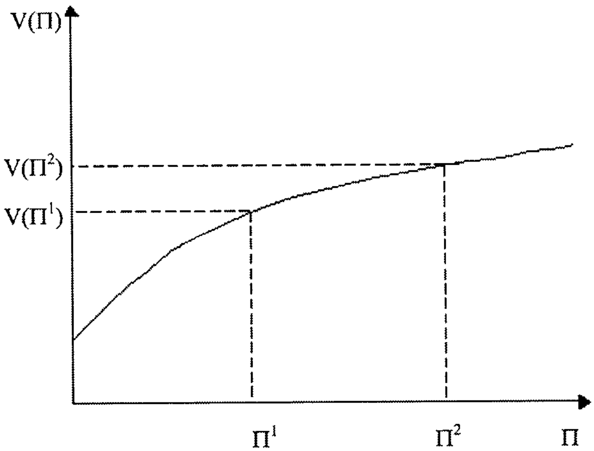

Figure 1.
Control limit policy

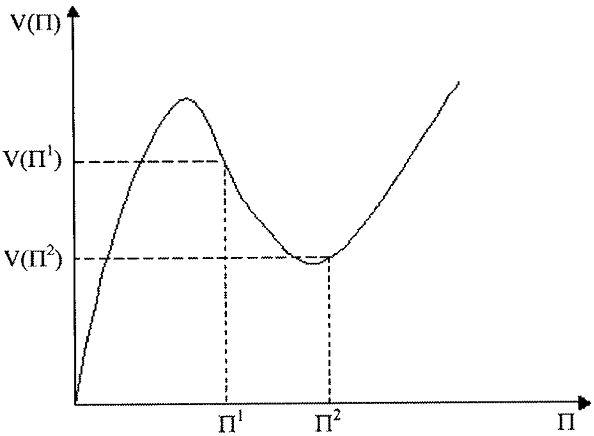

Figure 2.
Noncontrol limit policy

4.3 Lemmas
Let $\boldsymbol{\theta}_{-k}$ denote the outcome $\boldsymbol{\theta}=(\theta_1, \ldots, \theta_k, \ldots, \theta_L)$ in which all elements except θ_k are fixed, and $\boldsymbol{\Gamma}_{\boldsymbol{\theta}_{-k}}$ denotes the corresponding conditional probability matrix.

Lemma 4.1. For $\boldsymbol{\Pi}^1 <_T \boldsymbol{\Pi}^2$,

$$\sum_{i=1}^{n} \pi_i^1 g_i \leq \sum_{i=1}^{n} \pi_i^2 g_i \tag{9}$$

holds if g_i is a nondecreasing function of i.

Lemma 4.2. If $\boldsymbol{A}$ is a $(k_A \times k)$ TP_2 matrix and $\boldsymbol{B}$ is a $(k \times k_B)$ TP_2 matrix, $\boldsymbol{AB}$ is a $(k_A \times k_B)$ TP_2 matrix.

For proof, see Karlin (1968).

Lemma 4.3. If transition probability matrix $\boldsymbol{P} \in TP_2$,

$$\boldsymbol{\Pi}^1 \boldsymbol{P} <_T \boldsymbol{\Pi}^2 \boldsymbol{P} \quad \text{for } \boldsymbol{\Pi}^1 <_T \boldsymbol{\Pi}^2. \tag{10}$$

Lemma 4.4. For arbitrary $k \in (1, \ldots, L)$,

$$\Gamma \in \text{weak MLR} \Rightarrow \Gamma_{\boldsymbol{\theta}_{-k}} \in TP_2. \tag{11}$$

Proof. It is easy to prove this from the definitions of TP_2 and weak MLR. □

Lemma 4.5. If $\boldsymbol{P} \in TP_2$ and $\Gamma_{\boldsymbol{\theta}_{-k}} \in TP_2$,

$$P(\boldsymbol{\theta}_{-k}|\boldsymbol{\Pi}^1) <_T P(\boldsymbol{\theta}_{-k}|\boldsymbol{\Pi}^2) \quad \text{for } \boldsymbol{\Pi}^1 <_T \boldsymbol{\Pi}^2. \tag{12}$$

Proof. This is obvious from Lemmas 4.2 and 4.4 and the fact that $P(\boldsymbol{\theta}_{-k}|\boldsymbol{\Pi}) = \boldsymbol{\Pi P}\Gamma_{\boldsymbol{\theta}_{-k}}$. □

Lemma 4.6. For any fixed $\boldsymbol{\Pi}$,

$$T(\boldsymbol{\Pi}, \boldsymbol{\theta}_{-k}) <_T T(\boldsymbol{\Pi}, \boldsymbol{\theta}'_{-k}) \quad \text{for } \boldsymbol{\theta}_{-k} \leq \boldsymbol{\theta}'_{-k}. \tag{13}$$

Proof. Given *A2.*, $\Gamma \in$ weak MLR, so

$$\begin{vmatrix} T_j(\boldsymbol{\Pi}, \boldsymbol{\theta}_{-k}) & T_j(\boldsymbol{\Pi}, \boldsymbol{\theta}'_{-k}) \\ T_{j'}(\boldsymbol{\Pi}, \boldsymbol{\theta}_{-k}) & T_{j'}(\boldsymbol{\Pi}, \boldsymbol{\theta}'_{-k}) \end{vmatrix} = \frac{(\boldsymbol{\Pi P})_j (\boldsymbol{\Pi P})_{j'}}{(\boldsymbol{\Pi P}\Gamma)_{\boldsymbol{\theta}_{-k}} (\boldsymbol{\Pi P}\Gamma)_{\boldsymbol{\theta}'_{-k}}} \begin{vmatrix} \gamma_{j\boldsymbol{\theta}_{-k}} & \gamma_{j\boldsymbol{\theta}'_{-k}} \\ \gamma_{j'\boldsymbol{\theta}_{-k}} & \gamma_{j'\boldsymbol{\theta}'_{-k}} \end{vmatrix} \geq 0 \tag{14}$$

holds from Lemma 4.4 for every $j < j'$, and $\boldsymbol{\theta}_{-k} \leq \boldsymbol{\theta}'_{-k}$. □

Lemma 4.7. For any fixed $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_L)$, we have

$$T(\boldsymbol{\Pi}^1, \boldsymbol{\theta}) <_T T(\boldsymbol{\Pi}^2, \boldsymbol{\theta}) \quad \text{for } \boldsymbol{\Pi}^1 <_T \boldsymbol{\Pi}^2. \tag{15}$$

Proof. Given *A1.*, $\boldsymbol{P} \in TP_2$, so

$$\begin{vmatrix} T_i(\boldsymbol{\Pi}^1, \boldsymbol{\theta}) & T_i(\boldsymbol{\Pi}^2, \boldsymbol{\theta}) \\ T_j(\boldsymbol{\Pi}^1, \boldsymbol{\theta}) & T_j(\boldsymbol{\Pi}^2, \boldsymbol{\theta}) \end{vmatrix} = \frac{\gamma_{i\boldsymbol{\theta}} \gamma_{j\boldsymbol{\theta}}}{(\boldsymbol{\Pi}^1 \boldsymbol{P}\Gamma)_{\boldsymbol{\theta}} (\boldsymbol{\Pi}^2 \boldsymbol{P}\Gamma)_{\boldsymbol{\theta}}} \begin{vmatrix} (\boldsymbol{\Pi}^1 \boldsymbol{P})_i & (\boldsymbol{\Pi}^2 \boldsymbol{P})_i \\ (\boldsymbol{\Pi}^1 \boldsymbol{P})_j & (\boldsymbol{\Pi}^2 \boldsymbol{P})_j \end{vmatrix} \geq 0 \tag{16}$$

holds from Lemma 4.3 for every $i < j$, and $\boldsymbol{\Pi}^1 <_T \boldsymbol{\Pi}^2$. □

Lemma 4.8. If $V(\boldsymbol{\Pi})$ is a concave function of $\boldsymbol{\Pi}$, then $P(\boldsymbol{\theta}|\boldsymbol{\Pi})V(\boldsymbol{T}(\boldsymbol{\Pi}, \boldsymbol{\theta}))$ is also a concave function for any fixed $\boldsymbol{\theta}$.

Proof. For a proof, see Ohnishi *et al.* (1986). □

4.4 Sufficient condition

Here we examine the properties of optimal expected total cost function $V(\boldsymbol{\Pi})$ under the assumptions given in Section 2.2.

Corresponding to equation (7), we consider the functions $V^{(N)}(\boldsymbol{\Pi})(N = 0, 1, 2, \ldots)$ defined inductively as

$$V^{(0)}(\boldsymbol{\Pi}) \triangleq 0, \tag{17}$$

$$V^{(N)}(\boldsymbol{\Pi}) \triangleq \min\{KV^{(N-1)}(\boldsymbol{\Pi}), IV^{(N-1)}(\boldsymbol{\Pi}), RV^{(N-1)}(\boldsymbol{\Pi})\}. \tag{18}$$

$V^{(N)}(\boldsymbol{\Pi})$ is interpreted as the optimal expected cost over N periods. From the standard argument of contraction mapping theory, $V^{(N)}(\boldsymbol{\Pi})$ must converge to $V(\boldsymbol{\Pi})$ as N tends to infinity. Let $\boldsymbol{F}$ denote the class of all concave functions that have a property of control limit policy. Given *A1-A6*, we have the following theorems.

Theorem 4.1. If $V^{(N-1)}(\boldsymbol{\Pi}) \in \boldsymbol{F}$, then $KV^{(N-1)}(\boldsymbol{\Pi}) \in \boldsymbol{F}$.

Proof.

(1) *Control limit policy.* For

$$KV^{(N-1)}(\boldsymbol{\Pi}) = \sum_{i=1}^{n} \pi_i C_i + \beta \sum_{\boldsymbol{\theta}_1=1}^{m_1} \cdots \sum_{\boldsymbol{\theta}_L=1}^{m_L} P(\boldsymbol{\theta}|\boldsymbol{\Pi}) V^{(N-1)}(\boldsymbol{T}(\boldsymbol{\Pi}, \boldsymbol{\theta})), \tag{19}$$

we obtain

$$\sum_{i=1}^{n} \pi_i^1 C_i \leq \sum_{i=1}^{n} \pi_i^2 C_i \quad \text{for } \boldsymbol{\Pi}^1 <_T \boldsymbol{\Pi}^2. \tag{20}$$

based on *A3* and Lemma 4.1. Since $V^{(N-1)}(\boldsymbol{\Pi})$ has a property of control limit policy, and $\boldsymbol{T}(\boldsymbol{\Pi}^1, \boldsymbol{\theta}) <_T \boldsymbol{T}(\boldsymbol{\Pi}^2, \boldsymbol{\theta})$ holds by Lemma 4.7, we obtain

$$V^{(N-1)}(T(\boldsymbol{\Pi}^1, \boldsymbol{\theta})) \leq V^{(N-1)}(T(\boldsymbol{\Pi}^2, \boldsymbol{\theta})) \quad \text{for } \boldsymbol{\Pi}^1 <_T \boldsymbol{\Pi}^2. \tag{21}$$

Furthermore, we have

$$V^{(N-1)}(\boldsymbol{T}(\boldsymbol{\Pi}, \boldsymbol{\theta}_{-k})) \leq V^{(N-1)}(\boldsymbol{T}(\boldsymbol{\Pi}, \boldsymbol{\theta}'_{-k})) \tag{22}$$

from Lemma 4.6 and

$$P(\boldsymbol{\theta}_{-k}|\boldsymbol{\Pi}^1) <_T P(\boldsymbol{\theta}_{-k}|\boldsymbol{\Pi}^2) \quad \text{for } \boldsymbol{\Pi}^1 <_T \boldsymbol{\Pi}^2 \tag{23}$$

from Lemma 4.5. Thus, we obtain

$$\sum_{\boldsymbol{\theta}_k=1}^{m_k} P(\boldsymbol{\theta}_{-k}|\boldsymbol{\Pi}^1) V^{(N-1)}(\boldsymbol{T}(\boldsymbol{\Pi}^1, \boldsymbol{\theta}_{-k})) \leq \sum_{\boldsymbol{\theta}_k=1}^{m_k} P(\boldsymbol{\theta}_{-k}|\boldsymbol{\Pi}^2) V^{(N-1)}(\boldsymbol{T}(\boldsymbol{\Pi}^1, \boldsymbol{\theta}_{-k})), \tag{24}$$

where $V^{(N-1)}(\boldsymbol{T}(\boldsymbol{\Pi}^1, \boldsymbol{\theta}_{-k}))$ is a nondecreasing function of $\boldsymbol{\theta}_k$ for $\boldsymbol{\Pi}^1$. Since equation (24) holds for any $k \in (1, \ldots, L)$, we obtain

$$\sum_{\theta_1=1}^{m_1}\cdots\sum_{\theta_L=1}^{m_L} P(\boldsymbol{\theta}|\boldsymbol{\Pi}^1)V^{(N-1)}(\boldsymbol{T}(\boldsymbol{\Pi}^1, \boldsymbol{\theta})) \leq \sum_{\theta_1=1}^{m_1}\cdots\sum_{\theta_L=1}^{m_L} P(\boldsymbol{\theta}|\boldsymbol{\Pi}^2)V^{(N-1)}(\boldsymbol{T}(\boldsymbol{\Pi}^1, \boldsymbol{\theta})). \tag{25}$$

Given equations (21) and (25), we have

$$\sum_{\theta_1=1}^{m_1}\cdots\sum_{\theta_L=1}^{m_L} P(\boldsymbol{\theta}|\boldsymbol{\Pi}^1)V^{(N-1)}(\boldsymbol{T}(\boldsymbol{\Pi}^1, \boldsymbol{\theta})) \leq \sum_{\theta_1=1}^{m_1}\cdots\sum_{\theta_L=1}^{m_L} P(\boldsymbol{\theta}|\boldsymbol{\Pi}^2)V^{(N-1)}(\boldsymbol{T}(\boldsymbol{\Pi}^2, \boldsymbol{\theta})). \tag{26}$$

From equations (20) and (26), we derive

$$KV^{(N-1)}(\boldsymbol{\Pi}^1) \leq KV^{(N-1)}(\boldsymbol{\Pi}^2) \quad \text{for } \boldsymbol{\Pi}^1 <_T \boldsymbol{\Pi}^2. \tag{27}$$

That is, $KV^{(N-1)}(\boldsymbol{\Pi})$ has a property of control limit policy.

(2) *Concave.* For notational convenience, we rewrite equation (19) as

$$KV^{(N-1)}(\boldsymbol{\Pi}) = \boldsymbol{\Pi C} + \beta\sum_{\boldsymbol{\theta}} P(\boldsymbol{\theta}|\boldsymbol{\Pi})V^{(N-1)}(T(\boldsymbol{\Pi}, \boldsymbol{\theta})). \tag{28}$$

Since $V^{(N-1)}(\boldsymbol{\Pi}) \in \boldsymbol{F}$,

$$\sum_{\boldsymbol{\theta}} P(\boldsymbol{\theta}|\boldsymbol{\Pi})V^{(N-1)}(T(\boldsymbol{\Pi}, \boldsymbol{\theta}))$$

is a concave function of $\boldsymbol{\Pi}$ from Lemma 4.8 and the Appendix. As $\boldsymbol{\Pi C}$ is a linear function of $\boldsymbol{\Pi}$, $KV^{(N-1)}(\boldsymbol{\Pi})$, the combination of $\boldsymbol{\Pi C}$ and

$$\sum_{\boldsymbol{\theta}} P(\boldsymbol{\theta}|\boldsymbol{\Pi})V^{(N-1)}(T(\boldsymbol{\Pi}, \boldsymbol{\theta}))$$

is a concave function of $\boldsymbol{\Pi}$.

□

Theorem 4.2. If $V^{(N-1)}(\boldsymbol{\Pi}) \in \boldsymbol{F}$, then $IV^{(N-1)}(\boldsymbol{\Pi}) \in \boldsymbol{F}$.

Proof. For

$$IV^{(N-1)}(\boldsymbol{\Pi}) = \sum_{i=1}^{n} \pi_i C_i + I + \beta\sum_{j=1}^{n}\sum_{i=1}^{n} \pi_i p_{ij} V^{(N-1)}(\mathrm{e}^j), \tag{29}$$

from *A3*, *A4*, and Lemma 4.1,

$$\sum_{i=1}^{n} \pi_i^1 C_i + I \leq \sum_{i=1}^{n} \pi_i^2 C_i + I \quad \text{for } \mathbf{\Pi}^1 <_T \mathbf{\Pi}^2. \tag{30}$$

From Lemma 4.3, we have

$$\mathbf{\Pi}^1 \boldsymbol{P} <_T \mathbf{\Pi}^2 \boldsymbol{P} \quad \text{for } \mathbf{\Pi}^1 <_T \mathbf{\Pi}^2. \tag{31}$$

Since $\mathrm{e}^i <_T \mathrm{e}^j$ for $i < j$, we obtain a nondecrease of $V^{(N-1)}(\mathrm{e}^j)$ for $j \in \{1, \ldots, n\}$. Then

$$\beta \sum_{j=1}^{n} \sum_{i=1}^{n} \pi_i^1 p_{ij} V^{(N-1)}(\mathrm{e}^j) \leq \beta \sum_{j=1}^{n} \sum_{i=1}^{n} \pi_i^2 p_{ij} V^{(N-1)}(\mathrm{e}^j) \quad \text{for } \mathbf{\Pi}^1 <_T \mathbf{\Pi}^2. \tag{32}$$

Thus, we can derive

$$IV^{(N-1)}(\mathbf{\Pi}^1) \leq IV^{(N-1)}(\mathbf{\Pi}^2) \quad \text{for } \mathbf{\Pi}^1 <_T \mathbf{\Pi}^2 \tag{33}$$

from equations (30) and (32).

Since $IV^{(N-1)}(\mathbf{\Pi})$ is a linear function of $\mathbf{\Pi}$, $IV^{(N-1)}(\mathbf{\Pi})$ is a concave function. □

Theorem 4.3. If $V^{(N-1)}(\mathbf{\Pi}) \in \boldsymbol{F}$, then $RV^{(N-1)}(\mathbf{\Pi}) \in \boldsymbol{F}$.

Proof. Since $R + \beta V^{(N-1)}(\mathrm{e}^1)$ is constant, $RV^{(N-1)}(\mathbf{\Pi})$ is a special concave function that has a property of control limit policy for a general N. □

Theorem 4.4. $V^{(N)}(\mathbf{\Pi})$ is a concave function that has a property of control limit policy.

Proof. We use an inductive method to prove Theorem 4.4. The proof proceeds in three steps:

(1) assume that $V^{(N-1)}(\mathbf{\Pi}^1) \leq V^{(N-1)}(\mathbf{\Pi}^2)$ holds for $\mathbf{\Pi}^1 <_T \mathbf{\Pi}^2$;

(2) $N = 0$, $V^{(0)}(\mathbf{\Pi}) = 0$ is a special case of concave functions that have a property of control limit policy; and

(3) N periods, the total expected cost function for N periods is

$$V^{(N)}(\mathbf{\Pi}) = \min\{KV^{(N-1)}(\mathbf{\Pi}), IV^{(N-1)}(\mathbf{\Pi}), RV^{(N-1)}(\mathbf{\Pi})\}. \tag{34}$$

From Theorems 4.1, 4.2 and 4.3 we have $V^{(N)}(\mathbf{\Pi}) \in \boldsymbol{F}$. This guarantees $V(\mathbf{\Pi}) \in \boldsymbol{F}$ since $V^{(N)}(\mathbf{\Pi})$ converges to $V(\mathbf{\Pi})$ as N tends to infinity. □

4.5 Structural properties and discussion

Theorem 4.5. There exists an optimal policy that has four action regions at most. That is, the optimal policy is determined by $\mathbf{\Pi}^1$, $\mathbf{\Pi}^2$ and $\mathbf{\Pi}^3(\mathrm{e}^1 <_T \mathbf{\Pi}^1 <_T \mathbf{\Pi}^2 <_T \mathbf{\Pi}^3 <_T \mathrm{e}^n)$, such that the system keeps operating for $[\mathrm{e}^1, \mathbf{\Pi}^1] \cup [\mathbf{\Pi}^2, \mathbf{\Pi}^3]$, inspects for $[\mathbf{\Pi}^1, \mathbf{\Pi}^2]$, and replaces for $[\mathbf{\Pi}^3, \mathrm{e}^n]$.

The optimal policy obtained in the above theorem is similar to the one initially introduced by Ross (1971). The optimal policy with four regions ({keep, inspect, keep, replace}) may be described graphically as shown in Figure 3. Since an optimal policy which is a sequence of optimal actions depends on the given costs, sometimes fewer

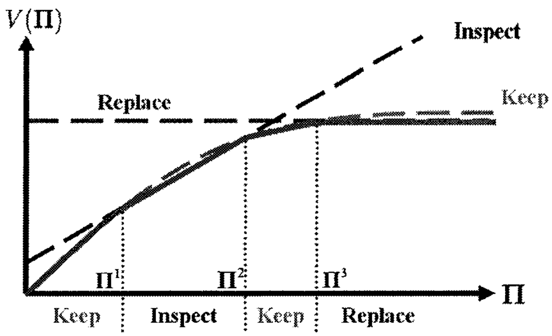

Figure 3. Four regions (keep, inspect, keep, replace)

than four regions can characterize the optimal policy under different cost functions. If the optimum policy is given by a control limit policy, we need only to determine at most three thresholds on which the optimal action changes. Subsequently we do not need to consider the optimal actions for each **Π**, so the calculation time to find an optimum procedure is reduced.

5. Conclusion

We investigated a deterioration system monitored by multiple, nonindependent monitors and found that the expected optimal cost function over an infinite horizon has a property of control limit policy given certain assumptions. Furthermore, the optimal policy has only four action regions at most, so the tremendous amount of calculation time required to find the optimum procedure can be reduced. This enables the best decision to be identified in a much shorter period of time.

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Appendix

- If $g(\mathbf{\Pi}, \boldsymbol{\theta})$ is a concave function of $\mathbf{\Pi}$ for each $\boldsymbol{\theta} \in \Theta$, $\Theta = \{\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^w\}$, then

$$\sum_{\theta \in \Theta} g(\mathbf{\Pi}, \boldsymbol{\theta})$$

is also a concave function of $\mathbf{\Pi}$.

Proof. For $\mathbf{\Pi} = \lambda \mathbf{\Pi}^1 + (1 - \lambda)\mathbf{\Pi}^2$, $(0 \leq \lambda \geq 1)$, since $g(\mathbf{\Pi}, \boldsymbol{\theta})$ is a concave function of $\mathbf{\Pi}$ for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, we have

$$g(\mathbf{\Pi}, \boldsymbol{\theta}^1) \geq \lambda g(\mathbf{\Pi}^1, \boldsymbol{\theta}^1) + (1 - \lambda) g(\mathbf{\Pi}^2, \boldsymbol{\theta}^1), \tag{35}$$

$$\vdots$$

$$g(\mathbf{\Pi}, \boldsymbol{\theta}^w) \geq \lambda g(\mathbf{\Pi}^1, \boldsymbol{\theta}^w) + (1 - \lambda) g(\mathbf{\Pi}^2, \boldsymbol{\theta}^w), \tag{36}$$

where $\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^w \in \boldsymbol{\Theta}$. Taking a summation, we obtain

$$\sum_{\theta \in \Theta} g(\mathbf{\Pi}, \boldsymbol{\theta}) \geq \lambda \sum_{\theta \in \Theta} g(\mathbf{\Pi}^1, \boldsymbol{\theta}) + (1 - \lambda) \sum_{\theta \in \Theta} g(\mathbf{\Pi}^2, \boldsymbol{\theta}); \tag{37}$$

that is,

$$\sum_{\theta \in \Theta} g(\mathbf{\Pi}, \boldsymbol{\theta})$$

is a concave function of $\mathbf{\Pi}$. □